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International Journal of Heat and Mass Transfer 44 (2001) 2801–2808

International Journal of HEAT and MASS TRANSFER

www.elsevier.com/locate/ijhmt

Solutions for the correction of temperature measurements based on beaded thermocouples

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Received 16 March 2000; received in revised form 17 October 2000

Abstract

Thermocouples are a practical and potentially accurate means for determining surface temperatures in many applications. Yet, significant errors can often arise during the usage of beaded thermocouples because of the effects of junction displacement, contact resistance, and stray heat transfer to the surrounding environment. To help offset these potentially significant sources of errors, Duhamel's integral in Laplace form was first used to relate the response of a surface thermocouple to the true substrate temperature. Once formulated, both exact and approximate Laplace inversion methods were used to derive closed-form corrective solutions. The resulting thermocouple correction curves show good agreement between themselves and existing analytical expressions and indicate the potentially deleterious effects of junction displacement, contact resistance, and the stray heat transfer to the surrounding environment.

1. Introduction

While many temperature measurement techniques have been developed and used over the years, surface mounted thermocouples still represent one of the most practical means of monitoring a wide range of temperatures. Unfortunately, the use of thermocouples is not without difficulties because of the deleterious influences of the mass of the thermocouple, the displacement of the junction (point of measure) from the substrate surface, contact resistance between the thermocouple and surface, as well as any stray heat transfer to the surrounding environment that can occur during a variety of industrial practices. To help offset the potentially significant errors that may result from any combination of these effects, a number of useful, albeit empirically based thermocouple corrective models and error minimizing steps have also been developed over the years [1–4]. While all of these methods are useful in alleviating and/or quantifying the sources of errors, they are not always practical and/or often require additional thermocouple calibration tests.

In a significant step towards determining the underlying temperature or forcing function, recent studies [5] were able to derive unit response functions (kernels) for a wide range of conditions that include the displacement of the junction from the surface, contact resistance between the thermocouple and substrate surface, as well as any stray heat transfer to the surrounding environment. This comprehensive study was also able to derive a limited number of solutions for the substrate forcing functions under step and linear temperature changes by using Duhamel's integral. While these solutions offered significant improvements to the resolution of errors without empirical constants, their utility is still limited to step and linear substrate temperature histories by the mathematics of inversion. As a remedy to these shortcomings, correction functions based on more versatile polynomials in time have been recently developed [6,7] for intrinsic thermocouples. The next important step is for the determination of practical corrective functions for the more commonly used beaded thermocouple.

The purpose of this paper is therefore, to expand the available set of correction functions to include beaded thermocouples complete with any combination of junc-

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0017-9310/01/$ - see front matter
PII: S0017-9310(00)00327-6

Nomenclature

A	thermal diffusivity ratio, D_{Tc}/D_{Sb}
B	contact resistance coefficient
Bi	Biot number, hr/k
D	thermal diffusivity
K	thermal conductivity ratio, k_{Tc}/k_{Sb}
$R(t)$	thermocouple response
x	non-dimension junction displacement, X/r
c	inversion constant ($c = 2$)
h_c	contact resistance coefficient
h_∞	environmental convective coefficient
k	thermal conductivity
r	thermocouple wire radius
s	Laplace variable
s^*	Gaver–Stehfest variable
t	non-dimension time, $D_{Sb}\tau/r^2$
X	junction displacement
$a_1 - a_3$	integral-order polynomial coefficients
$b_1 - b_3$	half-order polynomial coefficients
c_1	non-dimension constant, $\beta/(8/\pi^2 + \beta)$
C_2	non-dimension constant, $4/(8/\pi + \beta\pi)$
C_3	non-dimension constant, $C_2 - 1$
$\Delta T(t)$	substrate temperature forcing function
$\Phi(t)$	thermocouple unit response
$\Theta(t)$	recurring function defined by Eq. (28)
$\Omega(t)$	recurring function defined by Eq. (19)
β	non-dimension constant, $k/\sqrt{A}$
$\delta_{a1} - \delta_{a3}$	integral-order correction functions
$\delta_{b1} - \delta_{b3}$	half-order correction functions
ϕ	recurring function defined by Eq. (21)
τ	time

Subscripts

I	intrinsic
B	includes contact resistance
Bi	includes stray heat transfer
Sb	substrate
Tc	thermocouple
x	includes junction displacement

tion displacement, contact resistance, and stray heat transfer to the surrounding environment. To accomplish this and allow the accurate determination of surface temperatures, solutions for substrate forcing functions are derived by using direct Laplace inversion of a solution based on Duhamel's integral and a polynomial approximating the measured response. In addition, two approximate inversion techniques are used to derive closed-form solutions capable of modeling the response of a wide range of thermocouple types using measured temperature data.

2. Analytical considerations

The analysis begins with an idealization of a beaded thermocouple complete with the potential for junction displacement, contact resistance, and stray heat transfer as shown in Fig. 1. For this idealized thermocouple, a measurable response $R(t)$ will ensue after the substrate has undergone a time-dependent temperature change as dictated by an arbitrary forcing function $\Delta T(t)$. Duhamel's convolution integral [8] can then be used to relate the measured response, $R(t)$ and the unit response or kernel, $\Phi(t)$ to the unknown substrate forcing function:

$$R(t) = \frac{\partial}{\partial t}\int_0^t \Delta T(\xi)\cdot \Phi(t-\xi)\,\mathrm{d}\zeta \qquad (1)$$

provided the thermophysical properties are assumed to be independent of temperature. The significance of Eq. (1) arises from the fact that the only system information required to describe the response of the thermocouple to any forcing function is the unit response. However, this requires that a unit response containing all relevant terms for junction displacement, contact resistance, and stray heat transfer be fully defined.

Fortunately, a comprehensive, late time ($t > 0.1$) solution for the unit response has been derived [5] in the Laplace domain as

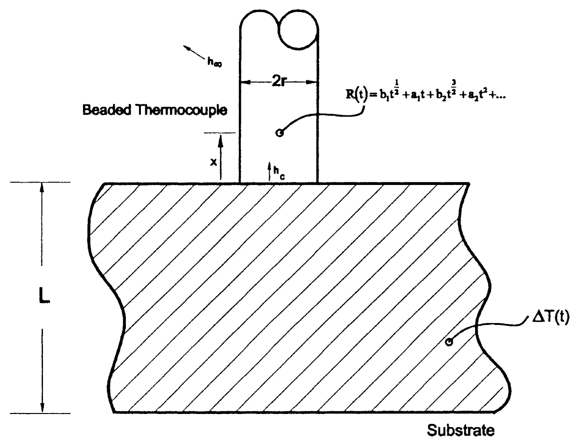

Fig. 1. Idealized thermocouple on a thermally thick substrate showing junction displacement x, contact resistance h_c, and stray heat transfer to the surrounding environment h_∞.

$$\Phi_{\mathrm{x,B,Bi}}(s) = \frac{\Psi_1(s)}{s\left[\Psi_1(s) + (K\Psi_2(s)/s\sqrt{A}) + (K\Psi_1(s)\Psi_2(s)/B\sqrt{A})\right]} \times \exp\left(\frac{-x\Psi_2(s)}{\sqrt{A}}\right), \tag{2}$$

where the recurring terms $\Psi_1(s)$ and $\Psi_2(s)$ are:

$$\Psi_1(s) = \frac{8}{\sqrt{s}\pi^2} + \frac{4}{s\pi}, \quad \Psi_2(s) = \sqrt{s + 4ABi} \tag{3}$$

where A is the non-dimensional diffusivity ratio, Bi the Biot modulus for heat transfer between the thermocouple and the external environment (with convective coefficient h_∞), and x is the non-dimensional junction displacement of the thermocouple relative to the substrate surface. These and a few additional non-dimensional relationships including the time variable, t are defined as follows:

$$A = \frac{D_{\mathrm{Sb}}}{D_{\mathrm{Tc}}}, \quad K = \frac{k_{\mathrm{Sb}}}{k_{\mathrm{Tc}}}, \quad B = \frac{h_{\mathrm{c}}r}{k_{\mathrm{Sb}}}, \quad Bi = \frac{h_\infty r}{k_{\mathrm{Tc}}},$$
$$\beta = \frac{K}{\sqrt{A}}, \quad t = \frac{D_{\mathrm{Sb}}\tau}{r^2}, \quad x = \frac{X}{r}. \tag{4}$$

In Eqs. (4), D is the thermal diffusivity (subscripts Sb and Tc are for the substrate and thermocouple, respectively), k the thermal conductivity, r the radius of the thermocouple wire, τ is dimensional time, and B is the contact Biot modulus (with contact coefficient h_{c}). All of the dimensional quantities and thermophysical properties can be measured directly. On the other hand, the terms describing the contact resistance can be estimated by using a variety of available methods [9–13]. Unfortunately, only certain limiting cases of the unit response defined by Eq. (2) may be obtained through direct inversion.

Once such case involved an intrinsic thermocouple where the junction is at the surface and there is no significant contact resistance or stray heat transfer ($x = 0, B = \infty$, and $Bi = 0$). The late time ($t > 0.1$) unit response, $\Phi(t)$ is defined as [5,14]:

$$\Phi(t) = 1 - C_1 \exp(C_2^2 t)\left[1 - \mathrm{erf}(C_2\sqrt{t})\right], \tag{5}$$

where

$$C_1 = \beta/(8/\pi^2 + \beta), \quad C_2 = 4/(8/\pi + \beta\pi). \tag{6}$$

For an ideal beaded thermocouple without contact resistance or heat transfer to the environment ($x > 0, B = \infty$, and $Bi = 0$), the late time unit response $\Phi_x(t)$, is also given as [5]:

$$\Phi_x(t,x) = \mathrm{erfc}\left(\frac{x}{2\sqrt{\mathrm{At}}}\right) - C_1 \exp\left(C_2\frac{x}{\sqrt{A}} + C_2^2 t\right) \times \mathrm{erfc}\left(\frac{x}{2\sqrt{\mathrm{At}}} + C_2\sqrt{t}\right), \tag{7}$$

where erfc(δ) represents the complimentary error function and all other variables are as previously defined.

Finally, for relatively large values of non-dimensional time ($t > 1$), the unit response for a beaded thermocouple where junction displacement, contact resistance, and stray heat transfer to the surrounding environment are considered ($x > 0, B \neq \infty$, and $Bi > 0$), can be approximated [5]:

$$\Phi_{\mathrm{x,B,Bi}}(t,x) \approx \frac{\exp\left(-2x\sqrt{Bi}\right)}{1 + 2K\sqrt{Bi}(1/B + \pi/4)} \left[1 + \frac{K\sqrt{Bi}}{\pi^2\sqrt{\pi t}} \times \frac{1}{1 + 2K\sqrt{Bi}(1/B + \pi/4)}\right]. \tag{8}$$

As would be expected, this relationship indicates that a residual steady-state error will always permeate the measured data because of the combined influence of the junction height and stray heat transfer to the surrounding environment.

3. Direct inversion

For the correction of temperature data measured using thermocouples, the utility of Eq. (2) is ultimately limited by its ability to be inverted and evaluated in the convolution integral in Eq. (1). However, an alternate approach that solves for the forcing temperature function $\Delta T(t)$, actually proves to be somewhat more tractable, even though direct inversion of a seemingly more complicated expression is required. Using Eq. (1), a generalized solution for the unknown temperature forcing function $\Delta T(t)$, may be realized by taking the Laplace transform of Eq. (1) and rearranging terms such that

$$\Delta T(s) = \frac{R(s)}{s \cdot \Phi_{\mathrm{x,B,Bi}}(s)}, \tag{9}$$

where $R(s)$ is an arbitrary expression representing the Laplace transform of the measured response of the thermocouple $R(t)$. For practical reasons, a response composed of integral- and half-order polynomials in time is suggested

$$R(t) = \sum_{j=1}^{3}\left[a_j t^j + b_j t^{j-(1/2)}\right], \tag{10}$$

such that the following Laplace form results:

$$R(s) = \sum_{j=1}^{3}\left[a_j \frac{\Gamma(j+1)}{s^{j+1}} + b_j \frac{\Gamma(j+1/2)}{s^{j+1/2}}\right], \tag{11}$$

where Γ represents the Gamma function and the coefficients a_j and b_j are for the integral- and half-order polynomial terms, respectively. It is important to note that the restriction of the series to six terms of integral- and half-order powers of time is arbitrary and does not represent a limitation of the method. However, the t^0 polynomial term must be avoided if the current form of Eq. (1) is to be used.

As shown in the generalized derivations, polynomials can be used to approximate the measured response of a thermocouple. The use of polynomials offers a number of advantages and potential drawbacks for the current analysis. In terms of advantages, the use of a versatile polynomial allows the approximation of a wide range of measured data. Moreover, properly used polynomials can help smooth out the errors that are inherent to any measurement process. As shown in Fig. 2, a relatively simple third-order polynomial can be used to smooth the measured temperature data. However, care must always be exercised with polynomials (or any approximating function) to ensure a reasonable fit to the data; lower order polynomials may not adequately fit or describe the data and transient trends while the addition of higher order terms may result in curve instability or "wiggle". Both types of curve inadequacies can introduce significant errors into the analysis that will be machine and algorithm dependent. Furthermore and irrespective of the type and order used, the approximate curve should never be used beyond the data interval used to define it. Finally, the measured data and what appear to be random errors may actually be changes in the surface temperature that reflect sudden variations of the underlying excitation. Unfortunately, this is a problem inherent to any inverse analysis and cannot be easily rectified without using an adaptive procedure that goes

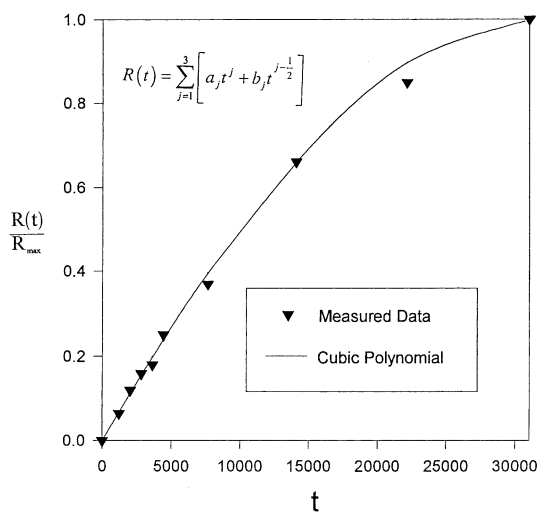

Fig. 2. Time-dependent response of 0.1mm diameter type-k beaded thermocouple showing the smoothed approximation from a third-order polynomial ($a_1 = 2.32\mathrm{E}-2$, $a_2 = -3.08\mathrm{E}-7$, $a_3 = -3.09\mathrm{E}-14$, and $b_1 = b_2 = b_3 = 0$).

beyond the intent of this paper to present straightforward analytical tools. Hence, engineering judgement should always be used to ensure that the polynomial adequately describes the data and does not ignore or exaggerate actual trends.

Using this general representation of $R(s)$ for the measured response, direct inversion of Eq. (9) yields the following expression for a substrate surface temperature or forcing function as defined by the polynomial coefficients:

$$\Delta T(t,x) = \sum_{j=1}^{3}\left[a_j\delta_{aj}(x,Bi,B,t) + b_j\delta_{bj}(x,Bi,B,t)\right], \quad t > 1, \tag{12}$$

where δ_{aj} and δ_{bj} represent the corrective functions. For beaded thermocouples with junction displacement x, contact resistance B, and stray heat transfer to the surrounding environment Bi, an approximate solution for the correction functions has been obtained using Eq. (8) for the unit response:

$$\delta_{a1}(x,Bi,B,t) = t + \frac{\theta^2}{\phi}\left[\pi - \frac{2\sqrt{t}}{\theta} - \frac{\pi\theta}{\sqrt{t}} + \sqrt{\pi}\Omega(t)\right], \tag{13}$$

$$\delta_{a2}(x,Bi,B,t) = t^2 + \frac{2}{\phi}\left[\pi^2\theta^4\left(1 - \frac{\theta}{\sqrt{t}}\right) + \pi\theta^2\left(t - 2\theta\sqrt{t}\right) - \frac{4}{3}\theta t^{3/2} + \pi^{3/2}\theta^4\Omega(t)\right], \tag{14}$$

$$\delta_{a3}(x,Bi,B,t) = t^3 + \frac{6}{\phi}\left[\pi^3\theta^6\left(1 - \frac{\theta}{\sqrt{t}}\right) + \pi^2\theta^4\left(t - 2\theta\sqrt{t}\right) - \frac{4}{3}\pi\theta^3 t^{3/2} + \frac{\pi\theta^2 t^2}{2} - \frac{8\theta t^{5/2}}{15} + \pi^{5/2}\theta^6\Omega(t)\right], \tag{15}$$

$$\delta_{b1}(x,Bi,B,t) = t^{1/2} + \frac{1}{2\phi}\left[\frac{\pi\theta^2}{2\sqrt{t}} - \pi\theta - \frac{\Omega(t)}{\theta}\right], \tag{16}$$

$$\delta_{b2}(x,Bi,B,t) = t^{3/2} + \frac{3}{4\phi}\left[\pi^2\theta^3\left(\frac{\theta}{\sqrt{t}} - 1\right) + \pi\theta\left(2\theta\sqrt{t} - t\right) - \pi^{3/2}\theta^3\Omega(t)\right], \tag{17}$$

$$\delta_{b2}(x,Bi,B,t) = t^{5/2} + \frac{5}{8\phi}\left[\pi^3\theta^5\left(\frac{\theta}{\sqrt{t}} - 1\right) + \pi^2\theta^3\left(2\theta\sqrt{t} - t\right) + \frac{5\pi\theta}{6\sqrt{t}} + \frac{8t^{5/2}}{15} - \pi^{5/2}\theta^5\Omega(t)\right], \tag{18}$$

with the recurring term $\Omega(t)$ containing the exponentially scaled, complimentary error function

$$\Omega(t) = \sqrt{\pi}\left[\frac{\theta}{\sqrt{t}} - \exp\left[\frac{t}{\pi\theta^2}\right]\mathrm{erfc}\left(\frac{\sqrt{t}}{\sqrt{\pi}\theta}\right)\right], \tag{19}$$

a steady-state error term

$$\theta = \left(\frac{kBi}{\pi^{5/2}}\right)\frac{1}{1+2K\sqrt{Bi}((1/B)+(\pi/4))}, \tag{20}$$

and

$$\phi = \frac{\exp\left(-2x\sqrt{Bi}\right)}{1+2K\sqrt{Bi}((1/B)+(\pi/4))}. \tag{21}$$

It is important to note that these equations are approximate and restricted to relatively large values of non-dimensional times. As shown by the plot in Fig. 3 based on the measured data shown in Fig. 2, the applicability of Eqs. (13)–(21) is limited to the late-time asymptotic portion of the response curve and does not cover the more crucial early times where the response lag is the greatest. The predicted response curves do however, indicate residual errors caused by the contact resistance and stray heat transfer terms (B and Bi) that will not abate as steady-state conditions are reached. One obvious shortcoming of the late-time approximation given by Eq. (8) is that the influence of the junction displacement disappears if $Bi = 0$.

This is in contrast to the expressions for intrinsic thermocouples and the same measured response (Fig. 2) where all errors vanish at steady state because of the absence of any stray heat transfer as shown in Fig. 4. For the intrinsic thermocouple, the late time ($t > 0.1$) corrective coefficients are listed below:

$$\delta_{a1}(t) = t + \frac{C_1}{C_2^2}\left[C_3 + \frac{C_3^2}{C_2\sqrt{\pi t}} + \frac{2C_2\sqrt{t}}{\sqrt{\pi}} - \frac{C_3^3}{C_2}\,\Theta(t)\right], \tag{22}$$

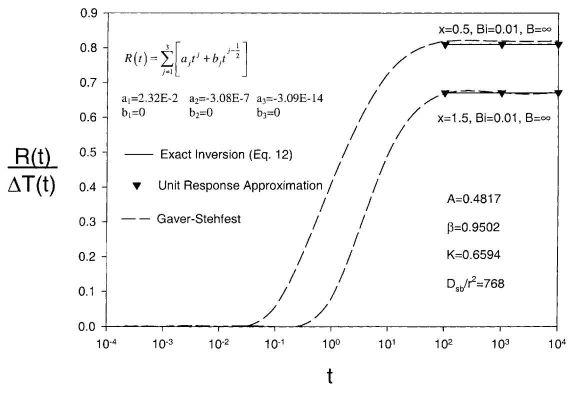

Fig. 3. Time-dependent response of a beaded thermocouple showing the effects of the junction displacement, stray heat transfer to the surrounding environment, and the resulting steady-state error.

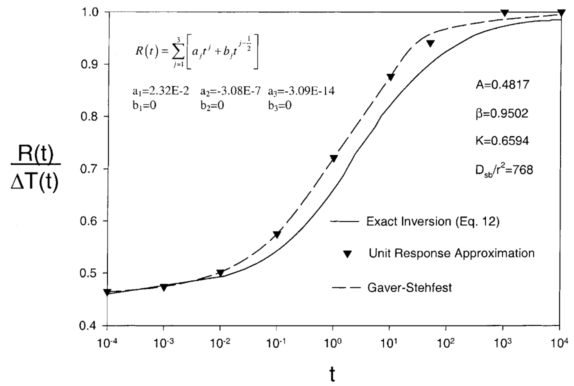

Fig. 4. Time-dependent response of an intrinsic thermocouple showing the absence of error once a steady-state condition has occurred.

$$\delta_{a2}(t) = t^2 + \frac{2C_1}{C_2^2}\left[\frac{C_3^3}{C_2^2} + C_3 t + \frac{C_3^4}{C_2^3\sqrt{\pi t}} + \frac{2C_3^2\sqrt{t}}{C_2\sqrt{\pi}} + \frac{4C_2 t^{3/2}}{3\sqrt{\pi}} - \frac{C_3^5}{C_2^3}\Theta(t)\right], \tag{23}$$

$$\delta_{a3}(t) = t^3 + \frac{6C_1}{C_2^2}\left[\frac{C_3^5}{C_2^4} + \frac{C_3^3 t}{C_2^2} + \frac{C_3 t^2}{2} + \frac{C_3^6}{C_2^5\sqrt{\pi t}} + \frac{2C_3^4\sqrt{t}}{C_2^3\sqrt{\pi}} + \frac{4C_3^3 t^{3/2}}{3C_2\sqrt{\pi}} + \frac{16C_2 t^{5/2}}{30\sqrt{\pi}} - \frac{C_3^7}{C_2^5}\Theta(t)\right], \tag{24}$$

$$\delta_{b1}(t) = \sqrt{t} + \frac{C_1\sqrt{\pi}}{2C_2^2}\left[\mathrm{C}_2 + \frac{C_3}{\sqrt{\pi t}} - C_3^2\Theta(t)\right], \tag{25}$$

$$\delta_{b2}(t) = t^{3/2} + \frac{2C_1\sqrt{\pi}}{4C_2^2}\left[\frac{C_3^2}{C_2} + C_2 t + \frac{C_3^3}{C_2^2\sqrt{\pi t}} + \frac{C_3\sqrt{t}}{\sqrt{\pi}} - \frac{C_3^4}{C_2^2}\Theta(t)\right], \tag{26}$$

$$\delta_{b3}(t) = t^{5/2} + \frac{15C_1\sqrt{\pi}}{8C_2^2}\left[\frac{C_3^4}{C_2^3} + \frac{C_3^2 t}{C_2} + \frac{t^2}{2C_2} + \frac{C_3^5}{C_2^4\sqrt{\pi t}} + \frac{4C_3^3\sqrt{t}}{C_2^2\sqrt{\pi}} + \frac{8C_3 t^{3/2}}{3\sqrt{\pi}} - \frac{C_3^6}{C_2^4}\Theta(t)\right] \tag{27}$$

and the recurring term $\Theta(t)$ can be simplified to contain the exponentially scaled, complimentary error function

$$\Theta(t) = \exp\left(\frac{C_2^2 t}{C_3^2}\right)\left[\mathrm{erfc}\left(\frac{C_2\sqrt{t}}{|C_3|}\right)\right] + \frac{1}{C_3\sqrt{\pi t}}$$

provided $C_3 = C_1 - 1$. (28)

4. Unit response approximation

As shown by the relationships in Eqs. (9)–(28), the correction of the various errors associated with the use of thermocouples collapses to the ability to describe the measured response with a function and invert the resulting expression. This is not a trivial problem as the required inversions of Eq. (2) or any expressions containing it, may not exist in closed-form [15]. Although numerical methods are always an option, they do little to help gain insight into the underlying physics. Moreover, the complicated nature of the expressions may cause convergence problems over a wide range of time, even if the most robust numerical routines are used.

As a remedy to the inversion problems just discussed, Direct and Indirect Laplace rules [16,17] can be used to derive an approximate solution for the corrected response of a beaded thermocouple. For this approximate approach that is limited to relatively monotonic functions of time, a Direct rule is first applied to determine the Laplace transform of an arbitrary monotonic function $\psi(t) \rightarrow \psi^*(s)$

$$\psi^*(s) = \left[\frac{1}{s}\psi(t)\right]_{t=(cs)^{-1}}, \tag{29}$$

where the constant $c = 2$. The inverse of the arbitrary function $\psi^*(s) \rightarrow \psi(t)$ may then be obtained by using a similar Inverse rule such that

$$\psi(t) = [s\psi^*(s)]_{s=(ct)^{-1}}. \tag{30}$$

Although neither of these relationships are exact from a mathematical standpoint, they are consistent with each other and are capable of producing reasonable results in certain instances. When the two rules are applied to an expression in the form of Eq. (9)

$$\Delta T(s) = \frac{R(s)}{s \cdot \Phi_{\mathrm{x,B,Bi}}(s)}, \tag{9}$$

a very interesting and useful result occurs in that the constant disappears and the expression simplifies to the original ratio of the measured response and unit response [18]

$$\Delta T(t) \approx \frac{R(t)}{\Phi_{\mathrm{x,B,Bi}}(t)}. \tag{31}$$

Hence, the unit response is the only system information required for the correction of temperature data and the estimation of the substrate forcing function. However, the measured response, $R(t)$ and underlying excitation $\Delta T(t)$ must be relatively monotonic for the approximation to be valid. Furthermore, the underlying excitation can be determined without any consideration of the accuracy of the function used to approximate $R(t)$. Accordingly, care must be exercised because the reasonableness of the approximation and smoothing of $R(t)$ will have a direct bearing on the accuracy of the prediction of $\Delta T(t)$.

As shown by the resulting curves in Fig. 3, excellent agreement was observed between the unit response approximation (URA) and the direct inversion given by Eq. (12). Although not verified analytically beyond the limited range, the correction curves predicted by the URA show the expected "quickening" of the thermocouple response as the junction height is decreased. In addition, the predicted response curves show the asymptotic approach towards the residual error caused by the combination of junction displacement and stray heat transfer to the environment. Fig. 4 shows a comparison of the URA to the exact solution for an intrinsic thermocouple. In this case, reasonable agreement is seen between the exact and approximate solutions over a wide range of non-dimensional time. Fig. 5 shows the URA response predictions for an ideal beaded thermocouple without contact resistance or stray heat transfer. As expected, without contact resistance and stray heat transfer to the surrounding environment, the errors will continuously diminish over time because the junction will eventually reach (and measure) the substrate temperature.

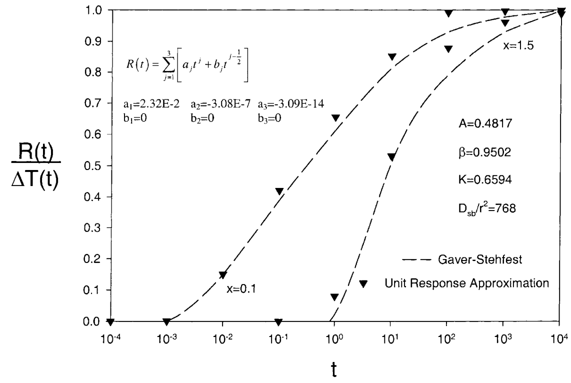

Fig. 5. Time-dependent response of an ideal beaded thermocouple showing the influence of junction displacement.

5. Gaver–Stehfest inversion

The URA appears to provide a reasonable estimation of the measurement errors associated with the use of intrinsic and beaded thermocouples. However, direct inversion of Eq. (2) in Laplace form and the resulting analytical solutions for the unit response may not always be possible. In these situations, an alternate inversion method [19–21] can be used

$$\psi(t) \approx \frac{\ln(2)}{t} \sum_{i=1}^{m} V_i \psi(s_i^*), \tag{32}$$

where the Laplace variable s, is modified

$$s_i^* = \frac{\ln(2)}{t} i \tag{33}$$

Table 1
First ten terms of the GS inverse approximation series

i	V_i
1	0.0833333333
2	−32.083333
3	1270.00076
4	−15,623.66689
5	84,244.16946
6	−236,957.5129
7	375,911.6923
8	−340,071.6923
9	164,062.5128
10	−32,812.50256

and the series coefficients V_i are defined through the following relationship:

$$V_i = (-1)^{i+m/2} \sum_{j}^{\min(i,m/2)} \times \frac{j^{m/2}(2\varphi)!}{((m/2)-j)!j!(j-1)!(i-j)!(2j-i)!}. \quad (34)$$

For the correct calculation of the coefficients, j must be restricted to the integer values of $(i+1)/2$ while m is the number of terms in the approximate inversion series that must always be kept even. For most functions, a 10-term expansion is usually sufficient; the coefficients, V_i, for the 10-term expansion used in this analysis are listed in Table 1. Otherwise, a 22-term expansion appears to be the upper limit when the functions are not monotonic. Increasing the number of terms in the series will not necessarily improve the accuracy or range of the method.

Using the resulting method denoted herein as GS (for the originators Gaver and Stehfest), the substrate forcing function can be readily expressed as the following multi-term series

$$\Delta T(t) \approx \frac{\ln(2)}{t} \sum_{i=1}^{m} V_i \frac{R(s_i^*)}{s_i^* \cdot \Phi_{\mathrm{x,B,Bi}}(s_i^*)}. \quad (35)$$

When the GS method was applied to the discussed thermocouple configurations with a response modeled by the cubic polynomial shown in Fig. 2, reasonable agreement was again observed. As shown by the superimposed curves in Figs. 3–5, the GS method provided a reasonable approximation to both the exact and URA solutions for the beaded and intrinsic thermocouples. Again, the correction curves show the expected quickening of the thermocouple response as the junction height is decreased, as well as the asymptotic approach towards the residual error caused by contact resistance and stray heat transfer to the environment. For the intrinsic thermocouple, as well as the ideal beaded thermocouple with junction displacement only, the errors continuously diminish over time as the junction eventually reaches the substrate temperature.

6. Conclusions

Both direct and approximate Laplace transform inversion methods were used to derive corrective solutions for intrinsic and beaded thermocouples that can be used to help correct measured temperature data. The use of direct inversion resulted in a late-time solution that reasonably predicts near steady-state behavior. An alternate, inverse Laplace transform method using approximate Direct and Inverse rules resulted in relatively simple corrective solutions based solely on the unit response. The resulting URA was shown to be capable of estimating thermocouple behavior over a wide range of time. However, the URA is limited by the availability and accuracy of unit response solutions and the monotonicity of the substrate temperature. To compensate for this limitation, a series approximation was also employed to generate corrective solutions for a wider variety of thermocouple types and loading conditions. For all of the explored thermocouple cases that include combinations of junction displacement, contact resistance between the bead and surface, and stray heat transfer to the surrounding environment, the resulting corrective expressions show reasonable agreement between themselves and exact expressions when available. In addition, the solutions highlight the potential significance of these errors, especially when stray heat transfer conditions are present.

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